MULTI-MOMENT THEORY OF EQUILIBRIUM OF THICK PLATES

PMM Vol. 32, No. 2, 1968, pp. 344-352

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(Received July 10, 1967)

One of the auhtors [1] gave a method of obtaining the differential equations and boundary conditions in problems of tension and bending of plates of constant thickness, on the basis of utilization of the minimum potential energy principle in combination with a symbolic writing of the solutions of the elasticity theory equations proposed by Lur'e [2 to 4]. The differential equations and geometric boundary conditions were obtained therein in general form in a natural way; however, to obtain the force boundary conditions required carrying out a great deal of awkward computations, which increased sharply in each successive approximation; the question of obtaining such boundary conditions in general form remained open.

These difficulties are overcome below; integration of the variation in the strain potential energy of the plate, through the thickness of the plate, and introduction of multi-moment state of stress characteristics substantially simplified the analysis and permittet both the geometric and static boundary conditions to be obtained in general form.

The displacements of points of the plate may be expressed in terms of six functions of the x,y coordinates which are the displacements u_0, v_0, w_0 of points of the middle plane of the plate and u'_0, v'_0, w'_0 the 'rotations'. Lur'e gave these expressions in symbolic form by using differentiation operators

$$\frac{\sin zD}{D}, \qquad \cos zD, \qquad D^2 = \Delta = \partial_1^2 + \partial_2^2, \qquad \partial_1 = \frac{\partial}{\partial x}, \qquad \partial_2 = \frac{\partial}{\partial y} \tag{0.1}$$

Expanding the symbolic operators in power series we obtain

$$u = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n} \Delta^n}{(2n)!} u_0 - \frac{m}{2(m-2)} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+2} \Delta^n}{(2n+1)!} \partial_1 \vartheta_0 +$$
$$+ \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1} \Delta^n}{(2n+1)!} u_0' - \frac{m}{4(m-1)} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+3} \Delta^n}{(2n+1)!\,(2n+3)} \partial_1 \vartheta_0' \tag{0.2}$$
$$w = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1} \Delta^n}{(2n+1)!} w_0' + \frac{m}{2(m-2)} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+3} \Delta^{n+1}}{(2n+1)!\,(2n+3)} \vartheta_0 +$$
$$+ \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n} \Delta^n}{(2n)!} w_0 - \frac{m}{4(m-1)} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+2} \Delta^n}{(2n+1)!} \vartheta_0'$$
$$\vartheta_0 = \partial_1 u_0 + \partial_2 v_0 + w_0', \qquad \vartheta_0' = \partial_1 u_0' + \partial_2 v_0' - \Delta w_0$$

Here m is Poisson's ratio. The expression for v is obtained from u by replacing u, u'_0, ∂_1 and v_0, v'_0, ∂_2.

Following Lur'e, it is convenient to separate the problem of deformation of a thick plate into two independent problems: the extension of the slab determined by the unknown functions u_0, v_0, w'_0, and the bending of the plate described by the functions u'_0, v'_0, w_0.

Let $\mathbf{p}^+$ and $\mathbf{p}^-$, respectively, denote the external force vectors per unit area of the endface planes $z = h$, $z = -h$. The projections of these forces on the x, y, z coordinate axes, which cause extension of the plate, are represented by

$$\eta_x = p_x^+ + p_x^-, \quad \eta_y = p_y^+ + p_y^-, \quad \zeta = p_z^+ - p_z^-$$

and those which cause bending of the plate by Formulas

$$t_x = p_x^+ - p_x^-, \quad t_y = p_y^+ - p_y^-, \quad p = p_z^+ + p_z^-$$

The elementary work of all the external forces applied to the plate endfaces is determined by Expression

$$\delta A' = \iint_{(\Omega)} (\mathbf{p}^+ \cdot \delta \mathbf{u}^+ + \mathbf{p}^- \cdot \delta \mathbf{u}^-) dx dy \tag{0.3}$$

Here Ω is the plate plantform area, and $\mathbf{u}^+$ and $\mathbf{u}^-$ are displacement vectors of the endface planes of the plate, whose projections are evaluated by means of (0.2) with z replaced therein by h or $-h$, respectively.

1. Problem of extension of a plate. The variation of the specific strain potential energy of a plate is determined by Formula

$$\delta\pi = \sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \sigma_z \delta\varepsilon_z + \tau_{xy} \delta\gamma_{xy} + \tau_{yz} \delta\gamma_{yz} + \tau_{zx} \delta\gamma_{zx} \tag{1.1}$$

Let us express the strains in terms of the desired functions u_0, v_0, w'_0, for which we utilize the known relationships between the strains and the derivatives of the displacements in addition to (0.2); we then have

$$\begin{aligned}
\varepsilon_x &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n} \Delta^n}{(2n)!} \partial_1 u_0 - \frac{m}{2(m-2)} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+2} \Delta^n}{(2n+1)!} \partial_1^2 \vartheta_0 \\
\varepsilon_z &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n} \Delta^n}{(2n)!} \{ w_0' + \frac{m}{2(m-2)} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+2} \Delta^{n+1}}{(2n+1)!} \vartheta_0 \\
\gamma_{xy} &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n} \Delta^n}{(2n)!} (\partial_2 u_0 + \partial_1 v_0) - \frac{m}{m-2} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+2} \Delta^n}{(2n+1)!} \partial_1 \partial_2 \vartheta_0 \\
\gamma_{zx} &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1} \Delta^n}{(2n+1)!} (\partial_1 w_0' - \Delta u_0) - \frac{m}{m-2} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1} \Delta^n}{(2n)!} \partial_1 \vartheta_0
\end{aligned} \tag{1.2}$$

The strains ε_y and γ_{zy} are obtained from the strains ε_x and γ_{zx} by a corresponding change of letters.

To derive the variations in the extension potential energy of a plate, we vary (1.2), and substitute the result into (1.1), after which we integrate the obtained relationship through the plate thickness. It is hence convenient to introduce the following static and hyper-static stress characteristics

$$T_x^{(n)} = \frac{(-1)^n}{(2n)!} \int_{-h}^{h} \sigma_x z^{2n} dz$$

$$T_y^{(n)} = \frac{(-1)^n}{(2n)!} \int_{-h}^{h} \sigma_y z^{2n} dz, \qquad S^{(n)} = \frac{(-1)^n}{(2n)!} \int_{-h}^{h} \tau_{xy} z^{2n} dz \tag{1.3}$$

$$\Gamma_x^{(n)} = \frac{(-1)^n}{(2n+1)!} \int_{-h}^{h} \tau_{zx} z^{2n+1} dz, \qquad \Gamma_y^{(n)} = \frac{(-1)^n}{(2n+1)!} \int_{-h}^{h} \tau_{yz} z^{2n+1} dz$$

Here $T_x^{(0)}$, $T_y^{(0)}$ are tensile, $S^{(0)}$ the shear forces, and $T_x^{(n)}$, $T_y^{(n)}$, $S^{(n)}$ (for $n \geq 1$) their hyper-static analogs; $\Gamma_x^{(0)}$, $\Gamma_y^{(0)}$ are the bi-forces, $\Gamma_x^{(n)}$, $\Gamma_y^{(n)}$ (for $n \geq 1$) the higher order bi-forces.

Let us also introduce a notation for the integrals

$$\frac{(-1)^n}{(2n)!} \int_{-h}^{h} \sigma_z z^{2n} dz = Z_t^{(n)} \tag{1.4}$$

characterizing the distribution of the stress σ_z through the plate thickness.

Therefore, by using (1.3), (1.4) and (1.2), we obtain

$$\begin{aligned}
\int_{-h}^{h} \sigma_x \delta\varepsilon_x dz &= T_x^{(0)} \partial_1 \delta u_0 + \sum_{n=1}^{\infty} T_x^{(n)} \partial_1 \delta\chi_x^{(n)} \\
\int_{-h}^{h} \sigma_z \delta\varepsilon_z dz &= Z_t^{(0)} \delta w_0' + \sum_{n=1}^{\infty} Z_t^{(n)} \delta\varphi'^{(n)} \\
\int_{-h}^{h} \tau_{xy} \delta\gamma_{xy} dz &= S^{(0)} \delta (\partial_1 v_0 + \partial_2 u_0) + \sum_{n=1}^{\infty} S^{(n)} \delta (\partial_1 \chi_y^{(n)} + \partial_2 \chi_x^{(n)}) \\
\int_{-h}^{h} \tau_{xz} \delta\gamma_{zx} dz &= \Gamma_x^{(0)} \partial_1 \delta w_0' + \sum_{n=1}^{\infty} (\Gamma_x^{(n)} \partial_1 \delta\varphi^{(n)} - \Gamma_x^{(n-1)} \partial_2 \delta\chi_x^{(n)})
\end{aligned} \tag{1.5}$$

where the integrals for σ_y and τ_{yz} are obtained from the integrals for σ_x and τ_{zx} by corresponding changes in the letters and subscripts.

Generalized coordinates

$$\chi_x^{(n)} = \Delta^n u_0 + \frac{nm}{m-2} \partial_1 \Delta^{n-1} \vartheta_0, \qquad \chi_y^{(n)} = \Delta^n v_0 + \frac{nm}{m-2} \partial_2 \Delta^{n-1} \vartheta_0 \tag{1.6}$$

$$\varphi^{(n)} = \Delta^n w_0' - \frac{nm}{m-2} \Delta^n \vartheta_0$$

corresponding to the generalized forces (1.3) introduced above, have been inserted into the relationships (1.5). The first two quantities in (1.6) can be treated as projections of the vector $\chi^{(n)}$, located in the middle plane of the plate; let us also note that $\chi_x^{(0)} = u_0$, $\chi_y^{(0)} = v_0$, $\varphi^{(0)} = w_0'$.

Summing all integrals of the type (1.5), then integrating over the plate area Ω and utilizing formulas for the transformation from double integrals over the domain Ω to integrals over the contour L surrounding the domain Ω, we obtain the following expression for the variation in the tensile potential energy of the plate:

$$\delta\Pi_1 = \oint_{(L)} \Big\{ (\nu_x T_x^{(0)} + \nu_y S^{(0)}) \delta u_0 + (\nu_x S^{(0)} + \nu_y T_y^{(0)}) \delta v_0 + (\nu_x \Gamma_x^{(0)} + \nu_y \Gamma_y^{(0)}) \delta w_0' +$$

$$+ \sum_{n=1}^{\infty} [(\nu_x T_x^{(n)} + \nu_y S^{(n)}) \delta\chi_x^{(n)} + (\nu_x S^{(n)} + \nu_y T_y^{(n)}) \delta\chi_y^{(n)} +$$

$$+ (\nu_x \Gamma_x^{(n)} + \nu_y \Gamma_y^{(n)}) \delta\varphi^{(n)}]\Big\} ds - \iint_{(\Omega)} \Big\{(\partial_1 T_x^{(0)} + \partial_2 S^{(0)}) \delta u_0 +$$

$$+ (\partial_1 S^{(0)} + \partial_2 T_y^{(0)}) \delta v_0 + (\partial_1 \Gamma_x^{(0)} + \partial_2 \Gamma_y^{(0)} - Z_l^{(0)}) \delta w_0' +$$

$$+ \sum_{n=1}^{\infty} [(\partial_1 T_x^{(n)} + \partial_2 S^{(n)} + \Gamma_x^{(n-1)}) \delta\chi_x^{(n)} + (\partial_1 S^{(n)} + \partial_2 T_y^{(n)} + \Gamma_y^{(n-1)}) \delta\chi_y^{(n)} +$$

$$+ (\partial_1 \Gamma_x^{(n)} + \partial_2 \Gamma_y^{(n)} - Z_l^{(n)}) \delta\varphi^{(n)}]\Big\} dx\, dy \tag{1.7}$$

Let us turn to the evaluation of the work of the external forces acting on the plate endfaces, defined by (0.3); for the tensile strain of the plate we have $\delta u^+ = \delta u^-$, $\delta v^+ = \delta v^-$, $\delta w^+ = -\delta w^-$; then

$$\delta A_1 = \iint_{(\Omega)} \Big\{\eta_x \delta u_0 + \eta_y \delta v_0 + h\zeta \delta w_0' +$$

$$+ \sum_{n=1}^{\infty} \frac{(-1)^n h^{2n}}{(2n)!} \Big[(\eta_x \delta\chi_x^{(n)} + \eta_y \delta\chi_y^{(n)}) + \frac{h\zeta}{2n+1} \delta\varphi^{(n)}\Big]\Big\} dx\, dy \tag{1.8}$$

For the forces q_ν, acting on the lateral surface, the elementary works is (1.9)

$$\delta A_3 = \int_{-h}^{h} dz \oint_{(L)} \mathbf{q}_\nu \cdot \delta \mathbf{u}\, ds = \int_{-h}^{h} dz \oint_{(L)} (q_{\nu x} \delta u + q_{\nu y} \delta v + q_{\nu z} \delta w)\, ds. \tag{1.9}$$

We interchange the order of integration in (1.9) and we express the variations δu, δv, δw, by using (0.2). Integrating through the plate thickness, and utilizing the notation of [1] for the static and hyper-static characteristics of the lateral loading q_ν, we obtain

$$\delta A_3 = \oint_{(L)} \Big[R_x^{(0)} \delta u_0 + R_y^{(0)} \delta v_0 + W^{(0)} \delta w_0' +$$

$$+ \sum_{n=1}^{\infty} (R_x^{(n)} \delta\chi_x^{(n)} + R_y^{(n)} \delta\chi_y^{(n)} + W^{(n)} \delta\varphi^{(n)})\Big] ds \tag{1.10}$$

$$R_x^{(n)} = \frac{(-1)^n}{(2n)!} \int_{-h}^{h} q_{\nu x} z^{2n} dz, \qquad R_y^{(n)} = \frac{(-1)^n}{(2n)!} \int_{-h}^{h} q_{\nu y} z^{2n} dz]$$

$$W^{(n)} = \frac{(-1)^n}{(2n+1)!} \int_{-h}^{h} q_{\nu z} z^{2n+1} dz \tag{1.11}$$

Here $R_x^{(0)}$, $R_y^{(0)}$ are projections of the main lateral loading vector on the x and y axes, and $w^{(0)}$ is the bi-force due to the lateral loading.

Applying the minimum potential energy principle, we have (1.12)

$$\delta\Pi_1 - \delta A_1 - \delta A_3 = 0 \tag{1.12}$$

Substituting (1.7), (1.8), (1.10) into (1.12), we obtain

$$\oint\limits_{(L)} \Big\{(\nu_x T_x^{(0)} + \nu_y S^{(0)} - R_x^{(0)})\,\delta u_0 + (\nu_x S^{(0)} + \nu_y T_y^{(0)} - R_y^{(0)})\,\delta v_0 +$$

$$+ (\nu_x \Gamma_x^{(0)} + \nu_y \Gamma_y^{(0)} - W^{(0)})\,\delta w_0' + \sum_{n=1}^{\infty}\Big[(\nu_x T_x^{(n)} + \nu_y S^{(n)} - R_x^{(n)})\,\delta\chi_x^{(n)} +$$

$$+ (\nu_x S^{(n)} + \nu_y T_y^{(n)} - R_y^{(n)})\,\delta\chi_y^{(n)} + (\nu_x \Gamma_x^{(n)} + \nu_y \Gamma_y^{(n)} - W^{(n)})\,\delta\varphi^{(n)}\Big]\Big\}\,ds -$$

$$- \iint\limits_{(\Omega)} \Big\{(\partial_1 T_x^{(0)} + \partial_2 S^{(0)} + \eta_x)\,\delta u_0 + (\partial_1 S^{(0)} + \partial_2 T_y^{(0)} + \eta_y)\,\delta v_0 +$$

$$+ (\partial_1 \Gamma_x^{(0)} + \partial_2 \Gamma_y^{(0)} - Z_t^{(0)} + h\zeta)\,\delta w_0' + \sum_{n=1}^{\infty}\Big[\Big(\partial_1 T_x^{(n)} + \partial_2 S^{(n)} + \qquad (1.13)$$

$$+ \Gamma_x^{(n-1)} + \frac{(-1)^n h^{2n}}{(2n)!}\,\eta_x\Big(\,\delta\chi_x^{(n)} + \Big(\partial_1 S^{(n)} + \partial_2 T_y^{(n)} + \Gamma_y^{(n-1)} +$$

$$+ \frac{(-1)^n h^{2n}}{(2n)!}\,\eta_y\Big)\,\delta\chi_y^{(n)} + \Big(\partial_1 \Gamma_x^{(n)} + \partial_2 \Gamma_y^{(n)} - Z_t^{(n)} + \frac{(-1)^n h^{2n+1}}{(2n+1)!}\,\zeta\Big)\,\delta\varphi^{(n)}\Big]\Big\}\,dx\,dy$$

The expressions in the double integrals in the parentheses before the variations δu_0, δv_0, $\delta w_0'$, vanish, because they are the equilibrium equations. Using the connection between the stresses and displacements, as well as (0.2), we obtain after appropriate manipulation

$$\partial_1 T_x^{(0)} + \partial_2 S^{(0)} + \eta_x = \int_{-h}^{h}\Big(\partial_1 \sigma_x + \partial_2 \tau_{xy} + \frac{\partial \tau_{zx}}{\partial z}\Big)\,dz =$$

$$= \eta_x - 2\mu \sum_{n=0}^{\infty} \frac{(-1)^n h^{2n+1}}{(2n+1)!}\,\Delta^n\Big[\partial_1 w_0' - \Delta u_0 - (2n+1)\,\frac{m\partial_1\vartheta_0}{m-2}\Big] = 0$$

$$\partial_1 \Gamma_x^{(0)} + \partial_2 \Gamma_y^{(0)} - Z_t^{(0)} + \zeta h = \int_{-h}^{h}\Big(\partial_1 \tau_{zx} + \partial_2 \tau_{yz} + \frac{\partial \sigma_z}{\partial z}\Big)\,dz = \qquad (1.14)$$

$$= 4\mu \sum_{n=0}^{\infty} \frac{(-1)^n h^{2n+1}}{(2n)!}\,\Delta^n\Big(\frac{2nm-1}{m-2}\,\vartheta_0 - w_0'\Big) + h\zeta = 0$$

The coefficient of the variation δv_0 is obtained from the first formula in (1.14) by an appropriate substitution of letters and subscripts.

Let us show that the remaining parentheses for the variations $\delta\chi_x^{(n)}$, $\delta\chi_y^{(n)}$, $\delta\varphi^{(n)}$ in the double integral (1.13) also vanish. Indeed, performing analogous calculations, we have

$$\partial_1 T_x^{(n)} + \partial_2 S^{(n)} + \Gamma_x^{(n-1)} + \frac{(-1)^n h^{2n}}{(2n)!}\,\eta_x = \frac{(-1)^n}{(2n)!}\int_{-h}^{h}\Big(\partial_1 \sigma_x + \partial_2 \tau_{xy} + \frac{\partial \tau_{zx}}{\partial z}\Big)\,z^{2n}\,dz =$$

$$= \frac{(-1)^n h^{2n}}{(2n)!}\Big\{\eta_x - 2\mu \sum_{k=0}^{\infty} \frac{(-1)^k h^{2k+1}}{(2k+1)!}\,\Delta^k\Big[\partial_1 w_0' - \Delta u_0 - (2k+1)\,\frac{m\partial_1\vartheta_0}{m-2}\Big]\Big\} = 0$$

$$\partial_1 \Gamma_x^{(n)} + \partial_2 \Gamma_y^{(n)} - Z_t^{(n)} + \frac{(-1)^n h^{2n+1}}{(2n+1)!}\,\zeta = \frac{(-1)^n}{(2n+1)!}\int_{-h}^{h}\Big(\partial_1 \tau_{zx} + \partial_2 \tau_{yz} + \frac{\partial \sigma_z}{\partial z}\Big)\,z^{2n+1}\,dz =$$

$$= \frac{(-1)^n h^{2n+1}}{(2n+1)!}\Big\{4\mu \sum_{k=0}^{\infty} \frac{(-1)^k h^{2k}}{(2k)!}\,\Delta^k\Big[\frac{2km-1}{m-2}\,\vartheta_0 - w_0'\Big] + \zeta\Big\} = 0 \qquad (1.15)$$

i.e., we again obtain the equilibrium equations multiplied by powers of h.

The equilibrium equations for a thick plate, expressed in terms of variables connected with the middle plane, were first obtained by Lur'e [3] in 1942. As power series in the plate thickness these equations are the following for the plate extension problem

$$\sum_{n=0}^{\infty} \frac{(-1)^n h^{2n+1}}{(2n+1)!} \Delta^n \left[\partial_1 w_0' - \Delta u_0 - (2n+1)\frac{m\partial_1\vartheta_0}{m-2}\right] = \frac{\eta_x}{2\mu}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n h^{2n+1}}{(2n+1)!} \Delta^n \left[\partial_2 w_0' - \Delta v_0 - (2n+1)\frac{m\partial_2\vartheta_0}{m-2}\right] = \frac{\eta_y}{2\mu} \qquad (1.16)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n h^{2n}}{(2n)!} \Delta^n \left[w_0' - \frac{2nm-1}{m-2}\vartheta_0\right] = \frac{\zeta}{4\mu}$$

In order for the problem of plate extension to be formulated completely, it is necessary to pose three infinite sets of boundary conditions for the three infinite order differential Eqs. (1.16). This turns out to be realizable since the contour integral (1.13) contains a triple infinity of variations $\delta\chi_x^{(n)}$ $\delta\chi_y^{(n)}$, $\delta\varphi^{(n)}$ $(n = 0, 1, 2, \ldots)$. Thus, the geometric conditions for a clamped edge in a Cartesian coordinate system are

$$\chi_x^{(n)} = 0, \quad \chi_y^{(n)} = 0, \quad \varphi^{(n)} = 0 \quad (n = 0, 1, 2\ldots) \qquad (1.17)$$

where the quantities $\chi_x^{(n)}$, $\chi_y^{(n)}$, $\varphi^{(n)}$ are defined by (1.16).

The geometric conditions (1.17) were thus expressed earlier in [1]. The force boundary conditions

$$\nu_x T_x^{(n)} + \nu_y S^{(n)} = R_x^{(n)}, \quad \nu_x S^{(n)} + \nu_y T_y^{(n)} = R_y^{(n)}, \quad \nu_x \Gamma_x^{(n)} + \nu_y \Gamma_y^{(n)} = W^{(n)} \qquad (1.18)$$

also follow from the relationships (1.13).

Conditions (1.17) and (1.18) are expressed as projections on Cartesian coordinate axes. It is easy to write boundary conditions with reference to axes connected with the plate outline. To do this, we should use the following relationships

$$\chi_\nu^{(n)} = \nu_x\chi_x^{(n)} + \nu_y\chi_y^{(n)}, \quad T_\nu^{(n)} = \nu_x^2 T_x^{(n)} + 2\nu_x\nu_y S^{(n)} + \nu_y^2 T_y^{(n)} \qquad (1.19)$$

$$\chi_s^{(n)} = \nu_x\chi_y^{(n)} - \nu_y\chi_x^{(n)}, \quad S_{\nu s}^{(n)} = \nu_x\nu_y(T_y^{(n)} - T_x^{(n)}) + (\nu_x^2 - \nu_y^2) S^{(n)}$$

$$\Gamma_\nu^{(n)} = \nu_x\Gamma_x^{(n)} + \nu_y\Gamma_y^{(n)}, \quad R_\nu^{(n)} = \nu_x R_x^{(n)} + \nu_y^{(n)} R_y^{(n)}, \quad R_s^{(n)} = \nu_x R_y^{(n)} - \nu_y R_x^{(n)}$$

which are the customary transformation formulas for vector and tensor components for a rotation of the coordinate axes system. The contour integral of the variational relationship (1.13) is hence rewritten as

$$\oint_{(L)} \sum_{n=0}^{\infty} [(T_\nu^{(n)} - R_\nu^{(n)})\delta\chi_\nu^{(n)} + (S_{\nu s}^{(n)} - R_s^{(n)})\delta\chi_s^{(n)} + (\Gamma_\nu^{(n)} - W^{(n)})\delta\varphi^{(n)}]\, ds \qquad (1.20)$$

from which result the geometric conditions for a clamped edge

$$\chi_\nu^{(n)} = 0, \quad \chi_s^{(n)} = 0, \quad \varphi^{(n)} = 0 \quad (n = 0, 1, 2, \ldots) \qquad (1.21)$$

and the natural force conditions for a free edge

$$T_\nu^{(n)} = R_\nu^{(n)}, \qquad S_{\nu s}^{(n)} = R_s^{(n)}, \qquad \Gamma_\nu^{(n)} = W^{(n)} \qquad (1.22)$$

2. Bending Problem. The desired functions of the plate bending problem are u'_0, v'_0 and w_0. Evaluating the strains in terms of the displacements (0.2), we obtain

$$\begin{aligned}
\varepsilon_x &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}\Delta^n}{(2n+1)!}\partial_1 u_0' - \frac{m}{4(m-1)}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+3}\Delta^n}{(2n+1)!\,(2n+3)}\partial_1^2\vartheta_0' \\
\varepsilon_z &= -\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}\Delta^n}{(2n+1)!} w_0 - \frac{m}{4(m-1)}\sum_{n=0}^{\infty}\frac{(-1)^n (2n+2)\, z^{2n+1}\Delta^n}{(2n+1)!}\vartheta_0' \\
\gamma_{xy} &= \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}\Delta^n}{(2n+1)!}(\partial_1 v_0' + \partial_2 u_0') - \frac{m}{2(m-1)}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+3}\Delta^n}{(2n+1)!\,(2n+3)}\partial_1\partial_2\vartheta_0' \\
\gamma_{zx} &= \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}\Delta^n}{(2n)!}(u_0' + \partial_1 w_0) - \frac{m}{2(m-1)}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+3}\Delta^n}{(2n+1)!}\partial_1\vartheta_0'
\end{aligned} \tag{2.1}$$

The formulas for ε_y and γ_{yz} are obtained from those for ε_x and γ_{zx} by an appropriate change in the letters and subscripts.

Let us introduce static and hyper-static state of stress characteristics of the plate in bending

$$\begin{gathered}
G_x^{(n)} = \frac{(-1)^n}{(2n+1)!}\int_{-h}^{h}\sigma_x z^{2n+1}\,dz, \qquad G_y^{(n)} = \frac{(-1)^n}{(2n+1)!}\int_{-h}^{h}\sigma_y z^{2n+1}\,dz \\
H^{(n)} = \frac{(-1)^n}{(2n+1)!}\int_{-h}^{h}\tau_{xy} z^{2n+1}\,dz \\
N_x^{(n)} = \frac{(-1)^n}{(2n)!}\int_{-h}^{h}\tau_{zx} z^{2n}\,dz, \qquad N_y^{(n)} = \frac{(-1)^n}{(2n)!}\int_{-h}^{h}\tau_{yz} z^{2n}\,dz
\end{gathered} \tag{2.2}$$

Here $G_x^{(0)}$, $G_y^{(0)}$, $H^{(0)}$ are the bending moments and torque $N_x^{(0)}$, $N_y^{(0)}$ the transverse forces and $G_x^{(n)}$, $G_y^{(n)}$, $H^{(n)}$, $N_x^{(n)}$ and $N_y^{(n)}$ their hyper-static analogs. Let us also introduce integral distribution characteristics of the stress σ_z through the plate thickness

$$\frac{(-1)^{n+1}}{(2n+1)!}\int_{-h}^{h}\sigma_z z^{2n+1}\,dz = Z_f^{(n)} \tag{2.3}$$

After integration by parts, and reduction of similar terms in identical variations, the variation in bending potential energy of the plate can be represented as follows:

$$\begin{aligned}
\delta\Pi_2 = \oint_{(L)}\sum_{n=0}^{\infty}&[(\nu_x G_x^{(n)} + \nu_y H^{(n)})\,\delta\psi_x^{(n)} + (\nu_x H^{(n)} + \nu_y G_y^{(n)})\,\delta\psi_y^{(n)} + \\
&+ (\nu_x N_x^{(n)} + \nu_y N_y^{(n)})\,\delta\xi^{(n)}]\,ds - \iint_{(\Omega)}\{(\partial_1 G_x^{(0)} + \partial_2 H^{(0)} - N_x^{(0)})\,\delta u_0' + \\
&+ (\partial_1 H^{(0)} + \partial_2 G_y^{(0)} - N_y^{(0)})\,\delta v_0' + (\partial_1 N_x^{(0)} + \partial_2 N_y^{(0)})\,\delta w_0 +
\end{aligned}$$

$$+\sum_{n=1}^{\infty}[(\partial_1 G_x^{(n)}+\partial_2 H^{(n)}-N_x^{(n)})\,\delta\psi_x^{(n)}+(\partial_1 H^{(n)}+\partial_2 G_y^{(n)}-N_y^{(n)})\,\delta\psi_y^{(n)}+$$
$$+(\partial_1 N_x^{(n)}+\partial_2 N_y^{(n)}-Z_f^{(n-1)})\,\delta\xi^{(n)}]\}dx\,dy \qquad (2.4)$$

The following abbreviations (generalized coordinates) have been introduced here:

$$\psi_x^{(n)}=\Delta^n u_0'+\frac{nm}{2(m-1)}\,\partial_1\Delta^{n-1}\vartheta_0'$$
$$\psi_y^{(n)}=\Delta^n v_0'+\frac{nm}{2(m-1)}\,\partial_2\Delta^{n-1}\vartheta_0' \qquad (2.5)$$
$$\xi^{(n)}=\Delta^n w_0+\frac{nm}{2(m-1)}\,\Delta^{n-1}\vartheta_0'$$

The elementary work of the endface forces (0.3) is the following when expension (0.2) for the displacements is taken into account

$$\delta A_2=\iint_{(\Omega)}\Big\{ht_x\delta u_0'+ht_y\delta v_0'+p\delta w_0+\sum_{n=1}^{\infty}\frac{(-1)^n h^{2n}}{(2n)!}\times$$
$$\times\Big[p\delta\xi^{(n)}+\frac{h}{2n+1}(t_x\delta\psi_x^{(n)}+t_y\delta\psi_y^{(n)})\Big]\Big\}\,dx\,dy \qquad (2.6)$$

To evaluate the elementary work of forces applied to the lateral surface, we use the static and hyper-static integral characteristics of the lateral loading introduced in [1]:

$$M_x^{(n)}=-\frac{(-1)^n}{(2n+1)!}\int_{-h}^{h}q_{\nu y}z^{2n+1}\,dz$$
$$M_y^{(n)}=\frac{(-1)^n}{(2n+1)!}\int_{-h}^{h}q_{\nu x}z^{2n+1}\,dz,\quad Q^{(n)}=\frac{(-1)^n}{(2n)!}\int_{-h}^{h}q_{\nu z}z^{2n}dz \qquad (2.7)$$

Then

$$\delta A_4=\oint_{(L)}\sum_{n=0}^{\infty}(M_y^{(n)}\delta\psi_x^{(n)}-M_x^{(n)}\delta\psi_y^{(n)}+Q^{(n)}\delta\xi^{(n)})\,dS \qquad (2.8)$$

Substitution of (2.4), (2.6) and (2.8) into the principle of minimum system potential energy

$$\delta\Pi_2-\delta A_2-\delta A_4=0$$

yields

$$\oint_{(L)}\sum_{n=0}^{\infty}[(\nu_x G_x^{(n)}+\nu_y H^{(n)}-M_y^{(n)})\,\delta\psi_x^{(n)}+(\nu_x H^{(n)}+\nu_y G_y^{(n)}+M_x^{(n)})\,\delta\psi_y^{(n)}+$$
$$+(\nu_x N_x^{(n)}+\nu_y N_y^{(n)}-Q^{(n)})\,\delta\xi^{(n)}]\;ds-\iint_{(\Omega)}\Big\{\Big(\partial_1 G_x^{(0)}+\partial_2 H^{(0)}-N_x^{(0)}+$$
$$+ht_x\Big)\delta u_0'+(\partial_1 H^{(0)}+\partial_2 G_y^{(0)}-N_y^{(0)}+ht_y)\,\delta v_0'+(\partial_1 N_x^{(0)}+\partial_2 N_y^{(n)}+p)\,\delta w_0+$$

$$+\sum_{n=1}^{\infty}\Big[\Big(\partial_1 G_x^{(n)}+\partial_2 H^{(n)}-N_x^{(n)}+\frac{(-1)^n h^{2n+1}}{(2n+1)!}t_x\Big)\delta\psi_x^{(n)}+$$

$$+\Big(\partial_1 H^{(n)}+\partial_2 G_y^{(n)}-N_y^{(n)}+\frac{(-1)^n h^{2n+1}}{(2n+1)!}t_y\Big)\delta\psi_y^{(n)}+$$

$$+\Big(\partial_1 N_x^{(n)}+\partial_2 N_y^{(n)}-Z_j^{(n)}+\frac{(-1)^n h^{2n}}{(2n)!}p\Big)\delta\xi^{(n)}\Big]\Big\}\,dx\,dy=0. \qquad (2.9)$$

As in the extension problem, an analysis of the expressions in parentheses before the variations in the double integral in (2.9) leads to the three equilibrium equations written in the symbolic form of Lur'e [3]; these Eqs. are expressed in series as follows:

$$\sum_{n=0}^{\infty}\frac{(-1)^n h^{2n}}{(2n)!}\Delta^n\Big[\partial_1 w_0+u_0'+\frac{nm\partial_1\vartheta_0'}{(m-1)\Delta}\Big]=\frac{t_x}{2\mu}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n h^{2n}}{(2n)!}\Delta^n\Big[\partial_2 w_0+v_0'+\frac{nm\partial_2\vartheta_0'}{(m-1)\Delta}\Big]=\frac{t_y}{2\mu}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}h^{2n+1}}{(2n+1)!}\Delta^n\Big[\Delta w_0+\frac{nm+m-1}{2(m-1)}\vartheta_0'\Big]=\frac{p}{4\mu} \qquad (2.10)$$

The variational relationship (2.9) yields both geometric, and natural (force) boundary conditions in the bending problem of a thick plate. Conditions for rigid clamping of the edge of a thick plate (geometric conditions) are the following in Cartesian coordinates:

$$\psi_x^{(n)}=0,\quad \psi_y^{(n)}=0,\quad \xi^{(n)}=0\quad (n=0,1,2,\ldots) \qquad (2.11)$$

Natural boundary conditions for a free plate edge (force conditions) are obtained from the requirement that the coefficients of the variations in the generalized coordinates ($\delta\psi_x^{(n)}$, $\delta\psi_y^{n}$, $\delta\xi^{(n)}$) in the contour integral (2.9) vanish:

$$\nu_x G_x^{(n)}+\nu_y H^{(n)}=M_y^{(n)},\quad \nu_x H^{(n)}+\nu_y G_y^{(n)}=-M_x^{(n)},\quad \nu_x N_x^{(n)}+\nu_y N_y^{(n)}=Q^{(n)}$$
$$(n=0,1,2,\ldots) \qquad (2.12)$$

Conditions (2.11) and (2.12) are expressed in a Cartesian coordinate system. To transform to ν,s axes connected to the plate contour L in the contour integral (2.9), we should use the relationships

$$\psi_\nu^{(n)}=\nu_x\psi_x^{(n)}+\nu_y\psi_y^{(n)},\qquad \psi_s^{(n)}=\nu_x\psi_y^{(n)}-\nu_y\psi_x^{(n)}$$

$$G_\nu^{(n)}=\nu_x^2 G_x^{(n)}+2\nu_x\nu_y H^{(n)}+\nu_y^2 G_y^{(n)}$$

$$H_{\nu s}^{(n)}=\nu_x\nu_y(G_y^{(n)}-G_x^{(n)})+(\nu_x^2-\nu_y^2)H^{(n)} \qquad (2.13)$$

$$N_\nu^{(n)}=\nu_x N_x^{(n)}+\nu_y N_y^{(n)},\quad M_\nu^{(n)}=\nu_x M_y^{(n)}-\nu_y M_x^{(n)},\quad M_s^{(n)}=\nu_x M_x^{(n)}+\nu_y M_y^{(n)}$$

The contour integral (2.9) hence becomes

$$\oint_{(L)}\sum_{n=0}^{\infty}[(G_\nu^{(n)}-M_\nu^{(n)})\delta\psi_\nu^{(n)}+(H_{\nu s}^{(n)}+M_s^{(n)})\delta\psi_s^{(n)}+(N_\nu^{(n)}-Q^{(n)})\delta\xi^{(n)}]\,ds \qquad (2.14)$$

and the resulting boundary conditions are:

For a rigidly clamped edge

$$\psi_\nu^{(n)}=0,\qquad \psi_s^{(n)}=0,\qquad \xi^{(n)}=0 \qquad (2.15)$$

For a free edge

$$G_\nu^{(n)} = M_\nu^{(n)}, \qquad H_{\nu s}^{(n)} = -M_s^{(n)}, \qquad N_\nu^{(n)} = Q^{(n)} \tag{2.16}$$

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Translated by M.D.F.